OPTIMAL PRODUCTION PLANNING WHEN FINAL DEMAND IS STOCHASTIC AND INTER-RELATED

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I. INTRODUCTION: In a static input-output analytic framework, we predict the output vector (X_t) given the final consumption vector (C_t) for a future year (t) and the technical coefficient matrix (A_0) of the base year (0) assuming that A_0 would remain constant at the year t. Symbolically,

$$X_t = (I - A_0)^{-1} C_t \tag{I.1}$$

Alternatively, one may replace A_0 in (I.1) by extrapolated or updated technical coefficient matrix ($\hat{A}_t$). In a more refined procedure, dynamic system may be used to obtain

$$X_t = (I - \hat{A}_t - \hat{B}_t G)^{-1} C_t \tag{I.2}$$

Where $\hat{B}_t$ is the extrapolated capital coefficient matrix and G is the diagonal matrix of growth rates permissible in the economy.

In the procedures described above, C_t is exogenous while X_t is endogenous in nature. However, X_t, which represents the volume of output, also represents the amount of income generated in the economy and hence in turn generates demand for consumption. Viewed as such, C_t is not exogenous to the system.

There may be several possible procedures, each one suitable to a particular theoretical and institutional framework, to formulate the relationship between C_t and X_t, the latter determining the former. A simple procedure in line would be to consider employment as a function of the level of output and in turn, consumption demand as a function of the level of employment. Set to equilibrium, both C_t and X_t might be simultaneously estimated. One such procedure was elaborated by Leontief himself (Leontief, pp. 88-108).

As it has been observed, there may be two distinct procedures to predict X_t; one in which C_t is exogenous and the other in which C_t is endogenous. In the first case one may ask as to how to predict C_t since one must use some statistical or econometric method of forecasting, making C_t stochastic. In the second case, where C_t is endogenous, it has been observed that $C_t = f(N_t)$, where N refers to the number of persons employed. Irrespective of the fact whether the vector of final demand is an implicit or explicit function of N, the estimated values for the future must be stochastic. One must not overstress the possibility, or more exactly, the inevitability of stochasticity in the predicted economic quantities. It would be more so in case of the final demand as it depends on numerous factors, not all economic in nature and several of them unquantifiable. Moreover, one must note that the elements of the consumption vector, apart from being stochastic in nature, exhibit substitutability and complementarity as well. That is to say, that C_{it} (ith commodity) and C_{jt} (jth commodity) might well be correlated, strongly or weakly, positively or negatively, and this issue is relevant.

Thus the main issue to deal with in this paper is: given that C is stochastic and its elements are (contemporarily) correlated, how to determine the volume of the corresponding output, X.To pose this problem formally, it may be noted that to meet the final demand which is stochastic in nature, the decision of output vector runs a risk of over-production or under-production. Had there been no stochasticity in C, X could be determined at a level, say, $\bar{X}$. Now, C being stochastic, c = $\bar{C}$+σk, if we want to satisfy C at a probability level p. Here σ is the standard deviation of C and k is the factor associated with the probability level p such that C ≤ c. In such a situation, X must be determined such that the probability constraint on C is satisfied. There might be a number of possible choices of X that satisfy the probability constraint on C. However, our purpose is to decide X such that it minimizes over-production/under-production while meeting the probability constraint on C.

We propose to outline in the following sections the procedure of determining X under the condition of stochastic and correlated final demand elements. In section II, we discuss the possible strategies of defining the objective function to optimize. In section III, the statistical aspects of the method have been elaborated. Section IV deals with the issues regarding optimization. Section V presents the findings of some experiments. Finally, section VI probes into the possibilities of further generalization.

II. CHOICE OF THE OBJECTIVE FUNCTION: In the process of decision making regarding X under the stochastic conditions of C, the choice of objective function has a great deal of importance. If one envisages that X should be predicted such that it satisfies the stochastic demand C at a given level of probability p, one may plan to produce X such that

$$X = (I - A)^{-1} c \qquad \text{(II.1)}$$

Where C ≤ c and A = A_0 or $\hat{A}_t$ or $\hat{A}_t - \hat{B}_t G$.

But since c = $\bar{C}$+σk, we have

$$X = (I - A)^{-1} (\bar{C} + \sigma k) \qquad \text{(II.2)}$$

However, if we disregard stochasticity, for k set to zero, we have

$$\bar{X} = (I - A)^{-1} \bar{C} \qquad \text{(II.3)}$$

It is evident that $\bar{X}$ might be more or less than X both at the sector level and the gross level. Hence, one may define the objective function, Z

$$Z = \left| (\sum_i x_i) - (\sum_i \bar{x}_i) \right|^L \qquad \text{(II.4)}$$

or, alternatively,

$$Z = \sum_i \left| x_i - \bar{x}_i \right|^L \qquad \text{(II.5)}$$

where, i = 1,2,…,n and L is a positive integer. In the expression (II.4) above, Z has been defined by the distance at the gross level, while in (II.5) it has been defined by the distance at the sectoral level. Economic interpretations of (II.4) and (II.5) are different and so are their numerical solutions. If we allow for unrestricted substitutability between x_i and x_j then (II.4) will represent our intention more closely then would (II.5). Numerically also, if we minimize Z, min(Z) for (II.4) would be smaller than min(Z) for (II.5) in general.

The objective functions specified above may also be interpreted as loss functions. An economic justification for considering them as loss functions is that over-production may cause a glut and under-production may cause shortage. They may also lead to perverted substitution of demand for the goods of a particular industry in lieu of the goods of other industries, mal-functioning of employment and income generating mechanism, diseconomic utilization of resources and distorted price signals to the flow pattern of investible funds. These effects account for the loss of social welfare which may be represented by the objective function (II.4) or (II.5), economically interpreted as loss function to be minimized.

It is worthwhile to note that a loss function may be defined in a statistical sense as well. In this sense, covering the risk of over-production or under-production incurs a cost that increases progressively with an increase in the probability level at which the production decision is made. However, gains from covering the risk continue to decrease with an increase in the probability level chosen. Then, an optimal decision would be to cover the risk at the probability level, which minimizes the net loss. It is obvious that in this approach the choice of probability level is not arbitrary. Nevertheless, difficulties in defining loss function in a statistical sense might be considerably prohibitive and the classical approach to decision making stands to be the only resort.

III. STATISTICAL ASPECTS OF THE PROBLEM: Let any two variates C_1 and C_2 be normally distributed with means $\bar{C}_1$ and $\bar{C}_2$, standard deviations σ_1 and σ_2 (respectively) and the correlation coefficient ρ, and joint density function

$$f(C_1,C_2) = \{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}\}^{-1}\exp\{-\frac{1}{2}(1-\rho^2)^{-1}(Q_1^2 - 2\rho Q_1Q_2 + Q_2^2)\} \qquad \text{(III.1)}$$

where, $Q_i = (C_i - \bar{C}_i)/\sigma_i$; for i =1,2 and π =3.14592654 = 4tan^{-1}(1).

The density function (III.1) is constant on the ellipse

$$(1-\rho^2)^{-1}(Q_1^2 - 2\rho Q_1Q_2 + Q_2^2) = \kappa \qquad \text{(III.2)}$$

for every positive value of κ in a 2-dimensional Euclidean space.

Since we have defined $Q_i = (C_i - \bar{C}_i)/\sigma_i$, the center of the loci of constant density are at the origin defined in (III.2) above. The intercept on the Q_1 axis and Q_2 axis are equal. If ρ >0, the major axis of the ellipse is along the 45^0 line with a length of $2\{\kappa(1+\rho)\}^{1/2}$ and the minor axis has the length of $2\{\kappa(1-\rho)\}^{1/2}$. If ρ <0, the major axis of the ellipse is along the (90+45)^0 line with a length of $2\{\kappa(1-\rho)\}^{1/2}$ and the minor axis has the length of $2\{\kappa(1+\rho)\}^{1/2}$.

Thus, we may think of the density function (III.1) as a surface above the plane. The contours of equal density are contours of equal altitude on a topographical map; they indicate the shape of the probability surface (Anderson, p. 18). When we transform back to $C_i = \bar{C}_i + Q_i\sigma_i$, we expand each contour by a factor σ_i in the direction of the ith axis and shift the center to $(\bar{C}_1, \bar{C}_2)$. The numerical value of cumulative density function in the above case may be given by

$$F(c_1,c_2) = p(C_1 \le c_1, C_2 \le c_2) = p[\{(c_1 - \bar{C}_1)/\sigma_1\} \le Q_1, \{(c_2 - \bar{C}_2)/\sigma_2\} \le Q_2] \qquad \text{(III.3)}$$

Now, suppose we fix the value of Q_1 at some magnitude, say, q_1 and the value of probability in (III.3) at some magnitude (say, p=0.95), then q_2 (of Q_2) can be found out uniquely.

Geometrically, once we have fixed the altitude and one of the coordinates, the other coordinate can easily be known. Thus on a given iso-probability contour, one may find out as many pairs of Q_1 and Q_2 as one wishes, such that

$$p = \varphi(Q_1, Q_2) \quad \text{(III.4)}$$

where p is any given value of probability $0 \le p \le 1$. For any given value of p we have unique (q_1, q_2). Now, granted that for a given value of p we could find out (q_1, q_2), then we may transform them back to find out

$$C_i \le c_i = \bar{C}_i + Q_i\sigma_i \text{ for I = 1,2.} \quad \text{(III.5)}$$

which may be interpreted as: " at a probability level p the final demand C_i will be no greater than the magnitude $c_i = \bar{C}_i + Q_i\sigma_i$ or in other words, if we produce x_i such that it satisfies the final demand $c_i = \bar{C}_i + Q_i\sigma_i$ exactly, we will fail to fail to satisfy the stochastic final demand C_i with a probability (1-p).

IV. ISSUES IN OPTIMISATION: As we have observed in section III above, we may guarantee the satisfaction of the stochastic final demand C_i at a particular level p, if we produce x_i such that it satisfies $c_i = \bar{C}_i + Q_i\sigma_i$. However, since (III.4) holds, there might be an indefinite number of combinations of $Q_i = (C_i - \bar{C}_i)/\sigma_i$ which would satisfy the demand C_i at the probability level p. All such combinations may not be conformal to our objective function (II.4) or alternatively, (II.5). Naturally, we would like to choose that particular set of $Q_i = (C_i - \bar{C}_i)/\sigma_i$ which is conformal to our objective function to be optimized.

Moreover, when we take up the bivariate case, for a given pair (say #1) of final demand elements C_i and C_j distributed normally with means $\bar{C}_i$ and $\bar{C}_j$, standard deviations σ_i and σ_j and correlation ρ_{ij}, we have $p = \varphi_1(Q_i, Q_j)$. Similarly, for another pair (say, #2), we have $p = \varphi_2(Q_k, Q_l)$ and so on. Since they all form a system, Q_i and Q_j cannot be decided in isolation to Q_k and Q_l.

Then, given such m pairs, we decide Q_s such that our objective function is optimized. In this perspective, however, we must note that once p and Q_i (one of the coordinates on the ellipse) are given, Q_j (the other coordinate on the ellipse) is uniquely determined and thus, for optimizing our objective function the decision variables are only as many as the pairs of the correlated elements of the demand vector, that is, m in number. Denoting the index of such pairs by a new subscript we may write that our objective function

$$Z = g_p(X) \quad \text{(IV.1)}$$

And in turn

$$X = x_p(Q_{pi}, Q_{pj}); i, j = 1, 2, ..., m. \quad \text{(IV.2)}$$

and

$$Q_{pj} = Q_p(p, Q_{pi}) \qquad \text{(IV.3)}$$

Hence,

$$Z = g(Q_{pi}); i = 1, 2, ..., m. \qquad \text{(IV.4)}$$

In (IV.4) above, Q_{pi} may be decided by any suitable algorithm to yield an optimum value of Z.

V. SOME EXPERIMENTAL FINDINGS: To conduct an experiment with the idea developed above, we have taken up a 5 x 5 technical coefficient matrix of a hypothetical economy, and a final demand vector of the compatible order (table 1). We assume that the elements of the final demand vector are distributed with means and standard deviations as given in the table.

Table 1. Technical Coefficient Matrix and Stochastic Final Demand of a Hypothetical Economy

	Technical Coefficient Matrix					Final Demand	
sector	1	2	3	4	5	Mean	SD
1	0.050	0.080	0.090	0.120	0.160	200	6
2	0.030	0.010	0.080	0.210	0.160	300	12
3	0.090	0.080	0.010	0.210	0.070	400	10
4	0.160	0.090	0.180	0.030	0.270	350	6
5	0.290	0.057	0.089	0.150	0.017	409	0

First, we have solved for output (X) disregarding stochasicity, that is, assuming that $\overline{C}_i$ is the true C_i as in (I.1). Then we have resumed stochasticity of C_1, C_2, C_3 and C4. Note that C_5 is not stochastic and such an assumption does not result into any loss of generality. Further we have assumed that the first pair, C_1 and C_2, is correlated with ρ_{12} = -0.6 while the second pair, C_3 and C_4, is correlated with ρ_{34} = -0.55 and so on (for additional four different sets of values of ρ_{12} and ρ_{34} as explained in the footnote in table 2). For optimization we have assumed that our objective function as in (II.4) with L=1. We have optimized with p = 0.9025.

Optimal Solutions of Output with Stochastic Final Demand Vector

Sector	1	2	3	4	5	Total	p =0.9025; L=1
C	200.000	300.000	400.000	350.000	409.000	1659.000	Non-Stochastic
X	598.479	710.281	766.575	901.740	840.825	3817.900	C vector
C	212.762	284.285	414.551	338.670	409.000	1659.268	ρ_{12}= -0.60
X	611.358	694.687	779.745	893.990	843.730	3823.510	ρ_{34}= -0.55
C	209.549	317.718	415.288	359.792	409.000	1711.347	ρ_{12}= 0.75
X	618.746	738.309	792.316	926.367	854.518	3930.256	ρ_{34}= 0.60
C	212.750	315.587	414.382	338.481	409.000	1690.200	ρ_{12}= 0.70
X	616.212	728.855	784.374	899.945	848.471	3877.857	ρ_{34}= –0.60
C	209.607	317.988	415.068	359.543	409.000	1711.206	ρ_{12}= 0.70
X	618.746	738.483	792.038	926.057	854.456	3929.780	ρ_{34}= 0.70
C	209.906	318.194	415.672	359.683	409.000	1712.455	ρ_{12}= 0.60
X	619.305	738.944	792.873	926.588	854.804	3932.514	ρ_{34}= 0.55

Results obtained from FORTRAN program adapted by the author from HT Bates in Kuester and Mize, pp. 298-308.

It is interesting to note that for covering the risk of letting final demand go unsatisfied, one has to produce more (at the gross level) than what would have been produced otherwise (were the final demand vector non-stochastic). However, the extent to which one has to produce more depends upon the degree of correlation. One produces the largest quantity when ρ_{12} and ρ_{34} both were 0.70. The quantum of over-produce is the least when ρ_{12} = -0.6 and ρ_{34} = -0.55. At the sectoral level, when ρ_{ij} is negative, x_i is produced more if x_j is produced less (or vice versa). This is due to substitutability. On the other hand, positive ρ_{ij} leads to over-production of both x_i and x_j which is due to complementarity.

We may conclude, therefore, that in general, when we are faced with the problem of planning production to satisfy stochastic and inter-correlated final demand, we have to cover the risk by producing more than what we would have produced otherwise while uncertainty was disregarded. This incremental production may be regarded as the cost of meeting the uncertain demand. This cost might be more or less in magnitude which, in turn, would depend on the structure of inter-relationship among the elements of the demand vector and the level of probability chosen for satisfying them. Nonetheless, the technical coefficient matrix and its structure have their own role to play in this regard.

VI. POSSIBILITIES OF FURTHER GENERALISATION: We envisage generalization of the method proposed in this paper in three different directions:

(a) towards multivariate normal distribution,
(b) towards multivariate non-normal distribution, and
(c) towards an open economic structure in which the level of output stochastically determines the final demand. Below we discuss some preliminaries of these lines of generalization.

VI.1: Generalization to Multivariate Normal Distribution: In the multivariate normal distribution case, the density function may (analogous to III.1) be defined as:

$$f(C) = \{(2\pi)^n \left|\det(\Sigma)\right|\}^{-1/2}[\exp\{-\frac{1}{2}(C-\bar{C})'\Sigma^{-1}(C-\bar{C})\}] \qquad \text{(VI.1.1)}$$

where Σ is the variance-covariance matrix of C, $\det(\Sigma)$ is the determinant of the matrix in the argument and C is an n-dimensional random vector of final demand. In this scheme, our earlier bivariate multi-pair formulation had the matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} & 0 \\ 0 & 0 & \sigma_{43} & \sigma_{44} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{55} \end{bmatrix}$$

Since Σ was of the block-diagonal structure, we could go in further for an analysis that did not require 'simultaneous' treatment and we could continue with the bivariate strategy. However, if we think of the matrix Σ as

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{44} & \sigma_{45} & 0 \\ 0 & 0 & 0 & \sigma_{54} & \sigma_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{66} \end{bmatrix}$$

then, in the first block we have to take up a 'trivariate' structure while in the second block it would be a bivariate one. In case of tivariate strategy, we will have to find

$$F(C_1, C_2, C_3) = p(C_1 \le c_1, C_2 \le c_2, C_3 \le c_3)$$
$$= p[\{(C_1 - \overline{C}_1)/\sigma_1\} \le Q_1, \{(C_2 - \overline{C}_2)/\sigma_2\} \le Q_2, \{(C_3 - \overline{C}_3)/\sigma_3\} \le Q_3] \quad \text{(VI.1.2)}$$

Now, given the probability p and any two of (Q_1 , Q_2 , Q_3), the third one has to be found out such that the constraint of meeting the probability level is satisfied. Similarly, in the second block, any one of (Q_4 , Q_5) is to be given and the other has to be found out such that the probability constraint is satisfied. Thus, in all we have (2+1)=3 decision variables which will take part in optimization of the objective function.

To generalize, we may state that given the matrix Σ(n,n) decomposable into r blocks of s_r size independent of each other, each block will render $s_r - 1$ number of Q as decision variables in optimization of the objective function. The total number of decision variables, thus, would be = $\sum_{i=1}^{r} s_i - r$. It follows directly that if the variance-covariance matrix Σ is non-decomposable, it will have only one block of order n and hence the number of decision variables entering into the optimization strategy would be n-1.

Alternatively, it is possible to formulate the problem slightly differently as follows:

Min $Z = \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{x}_i \right|^L$ subject to constraints

$(p^* + e) \ge p(C_1 \le c_1, C_2 \le c_2, ..., C_n \le c_n)$ and $(p^* - e) \le p(C_1 \le c_1, C_2 \le c_2, ..., C_n \le c_n)$ (VI.1.3)

In the inequalities in (VI.1.3) above, p^* is the given probability (say 0.95) constraint to be satisfied within a permissible range of approximation (p-e, p+e). Now, $p(C_1 \le c_1, C_2 \le c_2, ..., C_n \le c_n)$ is the function of q_1, q_2 ,…,q_n , all q_s entering into the scheme of numerical integration. This formulation is different from the earlier one. Earlier, only $s_r - 1$ number of q_s were entering into the scheme of integration, while now, all s_r number of q_s are entering into the scheme. However, this reformulation is only algorithmically different and does not affect the conceptual outline of this paper.

VI.2. Generalization to Multivariate Non-normal Distribution: In general, since an arbitrary multivariate distribution does not render itself to be expressed in terms of parameters $\overline{C}$ and σ alone, we visualize the problem as follows:

The n jointly distributed variables $C_1, C_2, ..., C_n$ are said to be specified by a joint probability density function if there is a non-negative Borel function $f_c(.)$ such that for any Borel set B of n-tuple of real numbers the probability $p[(c_1, c_2, \dots, c_n)$ is in B] may be obtained (Parzen, pp. 194-197) by integrating $f_c(.)$ over B, or p_c [B] = $p[(c_1, c_2, \dots, c_n)$ is in B] given by

$$\iint_B \dots \int f_c(c_1^{'}, c_2^{'}, ..., c_n^{'}) \partial c_1^{'} \partial c_2^{'} ... \partial c_n^{'} \qquad \text{(VI.2.1)}$$

where $c_i^{'} \le c_i$ and thus,

$$p(C_1 \le c_1, C_2 \le c_2, ..., C_n \le c_n) = \int_{-\infty}^{c_1} \partial c_1^{'} \int_{-\infty}^{c_2} \partial c_2^{'} ... \int_{-\infty}^{c_n} \partial c_n^{'} f_c(c_1^{'}, c_2^{'}, ..., c_n^{'}) \qquad \text{(VI.2.2)}$$

Setting p in (VI.2.2) at a certain level and granting that for given $(c_1, c_2, \dots, c_{n-1})$ a unique value of c_n can be obtained, then n-1 decision variables may be used for optimization.

VI.3: Generalization to an Open Economic Structure: So far we have assumed that $\bar{C}$ (the mean level of C) is fixed and the level of output X is not affecting $\bar{C}$ in any way. However, if we grant that $\bar{C} = c_0(X_1, X_2, \dots, X_n)$, c_0 being the consumption function, and at certain level of probability we make use of the interval estimates of C, we must decide X such that it is consistent with the satisfaction of C at that given probability level. We can then recursively decide the levels of X and C together.

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